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Clustering of heavy particles in random self-similar flow

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A statistical description of heavy particles suspended in incompressible rough self-similar flows is developed. It is shown that, differently from smooth flows, particles do not form fractal clusters. They rather distribute inhomogeneously with a statistics that only depends on a local Stokes number, given by the ratio between the particles' response time and the turnover time associated with the observation scale. Particle clustering is reduced by the fluid roughness. Heuristic arguments supported by numerics explain this effect in terms of the algebraic tails of the probability density function of the velocity difference between two particles.

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Over the last decade important progress has been made in the study of tracers transported by turbulent flows. Tools borrowed from field theory, statistical physics, and the theory of random dynamical systems have opened the way to a unified understanding of the statistics and dynamics of such passively transported pointlike particles [1]. However, in most natural or industrial situations where one encounters particles suspended in a flow, the impurities have a finite size and a mass density different from that of the carrier fluid. The dynamics of such inertial particles differs markedly from that of simple tracers, and in particular, they form clusters where their interactions are strongly enhanced. The statistical description of such inhomogeneities in the case of turbulent carrier flows is of particular interest in engineering [2], cloud physics [3], and planetology [4].

Turbulence spans many active spatial and temporal scales. Most work on inertial particles has focused on describing their spatial distribution and, in particular, two-points statistics (see [5,6] and references therein) below the Kolmogorov scale, which is the smallest active length scale of the carrier flow. There the carrier velocity field is smooth and characterized by a single time scale. The finite response time of the inertial particles yields a dissipative dynamics, so that at such scales the particle trajectories converge toward a dynamically evolving attractor. For any given response time of the particles, their mass distribution is singular and generically scale invariant with multifractal properties [7–9]. With few exceptions [10–13], considerably less attention has been paid to particle dynamics above the Kolmogorov scale. There, the fluid velocity field is not smooth, but according to the Kolmogorov theory of 1941, self-similar with Hölder exponent $h=1/3$ [14]. Little is known about the basic mechanisms of clustering (and thus about the statistics of pair separation) at these scales. In particular, the theory of dynamical systems lacks the tools to tackle the nonsmoothness of the flow. The current state of knowledge can be summarized as follows. The finite response time of the suspended particles introduces a new scale. This breaks the self-similarity in the particle distribution, and clustering has a different origin from the smooth case [8]. This is consistent with the qualitative observation that particles typically have the largest deviation from uniformity when their response time is of the order of the eddy turnover time [11,15,16].

In this Rapid Communication we focus on the second-order statistics of the particle distribution at scales within the inertial range. These statistics can be completely described in terms of the pair separation dynamics. At these scales, two concurrent mechanisms responsible for clustering can be identified: a dissipative dynamics due to their viscous drag and ejection from persistent vortical regions by centrifugal forces [17]. In order to gain a systematic insight into clustering we focus only on the former by assuming δ correlation in time of the carrier flow: the absence of any persistent structure ensures that centrifugal forces play no role. Note that this model describes exactly the case of *very* heavy particles whose response time is much larger than the typical correlation time of the ambient fluid [18,19]. We show that (the scale invariance of the velocity field does not extend to the particle distribution, and that) clustering is weakened by the roughness of the carrier velocity. This behavior is traced back to the manner of how the roughness of the carrier flow affects the distribution of the particle relative velocity.

Within the considered model, the relative motion of two particles is described by the time evolution of their separation $\boldsymbol{R}$ [17,18]:

$$\tau\ddot{\boldsymbol{R}} = [\delta u(\boldsymbol{R},t) - \dot{\boldsymbol{R}}]. \qquad (1)$$

Overdots denote time derivatives, τ the particle response (Stokes) time, and $\delta\boldsymbol{u}(\boldsymbol{r},t)=\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)$ the fluid velocity difference. The velocity $\boldsymbol{u}$ is assumed to be a stationary, homogeneous, and isotropic Gaussian field with correlation

$$\langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x}',t')\rangle = [2D_0\delta_{ij} - B_{ij}(\boldsymbol{x}-\boldsymbol{x}')]\delta(t-t'), \qquad (2)$$

where D_0 is the velocity variance. For rough self-similar flows, the function B takes the form $B_{ij}(\boldsymbol{r}) = D_1 r^{2h}[(d-1+2h)\delta_{ij} - 2hr_ir_j/r^2]$, where $r=|\boldsymbol{r}|$, d is the space dimension, $h\in[0,1]$ the Hölder exponent of the carrier velocity field, and D_1 a constant measuring the turbulence intensity. This kind of velocity field was introduced by Kraichnan [20] to model passive scalar transport.

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By defining $s = t/\tau$ and rescaling $\boldsymbol{R}$ by the observation scale r, it is easily seen that the above dynamics, and thus all the statistical properties of particle pairs at scale r, only depends on the *local Stokes number* $\mathcal{S}(r) = D_1\tau/r^{2(1-h)}$. This dimensionless quantity, first introduced in [16], is the ratio between the particle response time τ and the turnover time at scale r. It measures the scale-dependent effects of inertia. At large scales ($r \to \infty$) inertia becomes negligible [$\mathcal{S}(r) \to 0$] and particles recover the incompressible dynamics of tracers. Conversely, since $\mathcal{S}(r) \to \infty$ for $r \to 0$, inertia effects dominate at small scales and the dynamics approaches that of free particles. For both $\mathcal{S}(r) \to 0$ and $\mathcal{S}(r) \to \infty$, the particles distribute uniformly in space, while strong inhomogeneities are expected for intermediate values of $\mathcal{S}(r)$. We impose reflective boundary conditions at $|\boldsymbol{R}| = L$ in order to assure stationarity of the statistics. Although the boundary conditions break self-similarity, the aforementioned scaling arguments apply for scales $\ll L$.

For smooth carrier flows ($h=1$), there is a unique time scale so that the dynamics only depends on the global Stokes number $\mathcal{S}(r) = \mathcal{S} = D_1\tau$. Inhomogeneities in the particle distribution can be quantified by the *correlation dimension* $\mathcal{D}_2$ given by

$$\mathcal{D}_2 = \lim_{r\to 0} \delta(r), \quad \delta(r) = d(\ln P_2(r))/d(\ln r), \tag{3}$$

were $P_2(r)$ denotes the probability that $|\boldsymbol{R}| < r$. In smooth δ-correlated flows, just as in real suspensions, the correlation dimension nontrivially depends on $\mathcal{S}$ [18].

For nonsmooth but Hölder-continuous flows, $\mathcal{D}_2 = d$ for *all* particle response times τ as $\mathcal{S}(r=0) = \infty$. However, information on the inhomogeneities of the particle distribution can be observed through the scale dependence of the local correlation dimension $\delta(r)$ defined in (3). Due to the self-similarity expected at scales $r \ll L$, $\delta(r)$ depends only on h and on $\mathcal{S}(r)$. This is confirmed numerically for $d=2$ in Fig. 1(a). From the figure we can deduce that with increasing roughness (decreasing h) clustering is weakening and the minimum of $\delta(r)$ gets closer to d. Notice that in the smooth case ($h=1$), $\mathcal{S}(r) = \mathcal{S}$ and the plotted data refer to the correlation dimension (see [18] for details).

We now turn to the typical velocity difference $\dot{\boldsymbol{R}}$ between two particles and its dependence on the separation $\boldsymbol{R}$. For smooth flows, when $|\boldsymbol{R}| \to 0$ an algebraic behavior of the form $|\dot{\boldsymbol{R}}| \sim |\boldsymbol{R}|^\gamma$ is observed, defining a Hölder exponent γ for the particle velocities. This exponent decreases from $\gamma = h = 1$ for $\mathcal{S} = 0$, corresponding to a differentiable particle velocity field, to $\gamma = 0$ for $\mathcal{S} \to \infty$, which means particles moving with uncorrelated velocities [18]. Similarly, in nonsmooth flows γ is asymptotically equal to the fluid Hölder exponent h at large scales [$\mathcal{S}(r) \to 0$] and approaches 0 at very small scales [$\mathcal{S}(r) \to \infty$]. Therefore, similarly to the case of $\delta(r)$, all relevant information appears in the scale dependence of the local exponent $\gamma(r)$ which should only depend on the fluid Hölder exponent and on the local Stokes number. This is confirmed by the collapse observed in Fig. 1(b), where the ratio $\gamma(r)/h$ is represented as a function of $\mathcal{S}(r)$ for various values of h. It is worth noticing that the

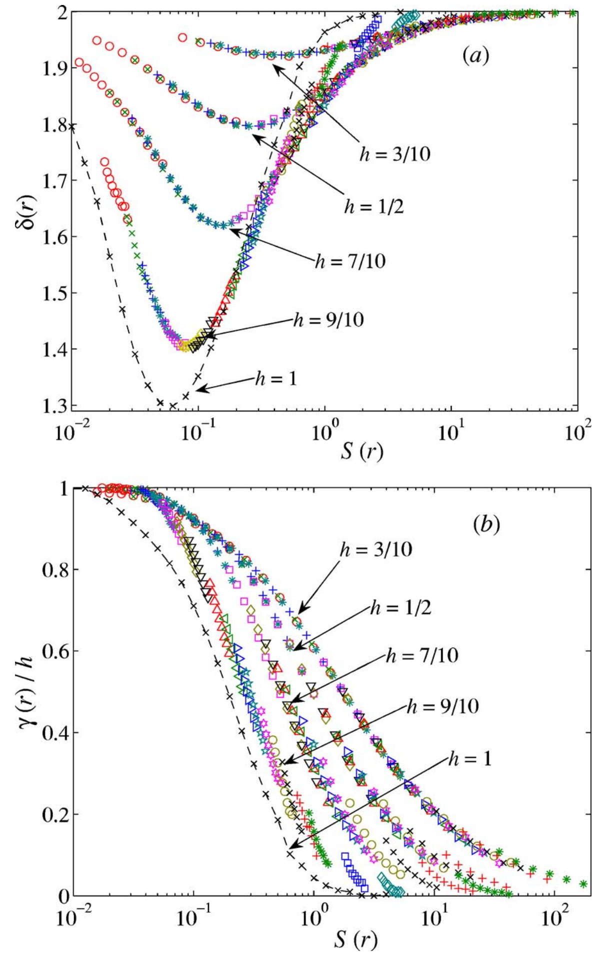

FIG. 1. (Color online) (a) Local correlation dimension $\delta(r)$ for various values of the particle response time τ (various symbols) and various scales r plotted as a function of the scale-dependent Stokes number $\mathcal{S}(r) = D_1\tau/r^{2(1-h)}$ for five values of the Hölder exponent h in two dimensions $d=2$. (b) Same for the ratio between the local exponent $\gamma(r)$ of the particle velocity and h.

transition from $\gamma(r) = h$ to $\gamma(r) = 0$ shifts towards larger values of the local Stokes number and broadens as h decreases. The fact that $\gamma(r) = h$ for $r \to \infty$ implies that the particles should asymptotically experience Richardson diffusion just as tracers.

For smooth flows, insight into the mechanisms of clustering is gained by considering the dynamics in terms of three variables only—the relative particle distance and the longitudinal and transversal velocity differences—instead of the full phase-space dynamics (1) and (2) [21,22]. Adapting this strategy to rough flows, the dynamics in $d=2$ is given by

$$\dot{X} = -X - Z^{-1}(hX^2 - Y^2) + \eta_1(s), \tag{4}$$

$$\dot{Y} = -Y - (1+h)Z^{-1}XY + \eta_2(s), \tag{5}$$

$$\dot{Z} = (1-h)X, \tag{6}$$

with X and Y referring to the longitudinal and transverse dimensionless velocity differences, respectively:

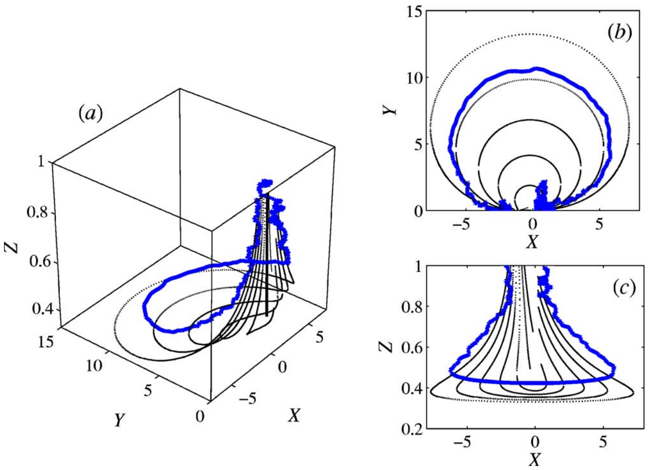

FIG. 2. (Color online) Phase-space picture of the system (4)–(6) for $h=7/10$. The thin smooth lines represent the drift. A random trajectory of the system with $\mathcal{S}(L)=1$ is shown in bold (blue online); it performs a large loop from $X<0$ to $X>0$. (a) The full (X,Y,Z) space, (b) and (c) projections in the $Z=0$ and $Y=0$ planes, respectively.

$$X = (\tau/L^2)(|\boldsymbol{R}|/L)^{-(1+h)}\boldsymbol{R}\cdot\dot{\boldsymbol{R}},$$

$$Y = (\tau/L^2)(|\boldsymbol{R}|/L)^{-(1+h)}|\boldsymbol{R}\times\dot{\boldsymbol{R}}|,$$

$$Z = (|\boldsymbol{R}|/L)^{1-h}, \tag{7}$$

The overdots now denote derivatives with respect to $s=t/\tau$; η_1 and η_2 are independent white noises with variances $2\mathcal{S}(L)$ and $2(1+2h)\mathcal{S}(L)$, respectively; $\mathcal{S}(L)=D_1\tau/L^{2(1-h)}$ is the Stokes number associated with the system size. Reflective boundary conditions at $|\boldsymbol{R}|=L$ in physical space imply reflective boundary conditions at $Z=1$; the peculiar form of the boundary conditions is expected to not change the properties at scales $\ll 1$. Y is ensured to remain positive by reflective boundary conditions at $Y=0$. Rescaling $|\boldsymbol{R}|$ with λ, and thus Z with λ^{1-h}, leads to transform X and Y to $\lambda^{1-h}X$ and $\lambda^{1-h}Y$ in order to confine the scaling factor in the noise. This again amounts to considering the same dynamics with a scale-dependent Stokes number $S(\lambda L)$. Equations (4)–(6) were used to produce the numerical results.

Figure 2 sketches the dynamics in the (X,Y,Z) space. The line $X=Y=0$ acts as a stable fixed line for the drift terms in Eqs. (4)–(6). A typical trajectory spends a long time diffusing around this line, until the noise realization becomes strong enough to escape from its neighborhood. When this happens with $X>0$, the quadratic terms in the drift drive the trajectory back to the stable line. On the contrary, if $X<0$ and $hX^2+XZ-Y^2<0$, the drift accelerates the trajectory towards larger negative values of X. Then the particles get closer to each other—i.e., Z decreases—until the quadratic terms in Eqs. (4) and (5) become dominant. The trajectory then loops back in the (X,Y) plane, approaching the stable line from its right.

During these loops, X becomes very large negative, and hence by Eq. (6), Z or equivalently the interparticle distance R becomes substantially small. The loops are thus the basic mechanism for clustering. As we now show, their statistical signature is the presence of algebraic tails for the probability density function (PDF) of the dimensionless velocity differences X and Y. Similarly to the case of smooth flows [18] such power laws can be understood in terms of the cumulative probability $P^{<}(x)=\Pr(X<x)$ with $x\ll -1$. The latter can be estimated as the product of (i) the probability to start a sufficiently large loop that reaches values more negative than x and (ii) the fraction of time spent by the trajectory at $X<x$. Therefore we assume that within a distance of order unity from the line $X=Y=0$ the quadratic terms in the drift are negligible and X and Y are independent Ornstein-Uhlenbeck processes, while at larger distances only the quadratic terms contribute.

Within this simplified dynamics, a loop is initiated at a time s_0 for which $X_0=X(s_0)<-1$ and $Y_0=Y(s_0)\ll|X_0|$. If the trajectory evolves on a loop in the (X,Y) plane, both $|X(s)|$ and $Y(s)$ become very large. Let us denote by x^* the largest negative value of X attained by the trajectory. Reaching values smaller than $x\ll -1$ is clearly equivalent to $x^*<x$. Far from the stable line $X=Y=0$, the noise can be neglected and the deterministic part of the dynamics can be integrated explicitly. After some standard algebraic manipulations which are not detailed here, one obtains that $x^*\propto[X_0+Z_0]X_0^hY_0^{-h}$. Hence, in order to reach values smaller than x, the loop should start with $Y_0<|x|^{-1/h}$. The probability to initiate such a loop is thus given by the probability to exit the noise-dominated region with $Y_0<|x|^{-1/h}$. There, Y is approximately an Ornstein-Uhlenbeck process, independent of X and Z and with a reflective boundary condition at $Y=0$. Contribution (i) is thus $\propto|x|^{-1/h}$. For the second contribution (ii), the fraction of time spent at $X<x$ can be obtained from the explicit form of the solution when the noise is neglected; it is also found to be $\propto|x|^{-1/h}$. Put together, the two contributions give $P^{<}(x)\propto|x|^{-2/h}$ when $x\ll -1$. Hence the PDF of the longitudinal velocity difference X has a power-law tail $p(x)=dP^{<}(x)/dx\propto|x|^{-\alpha}$ with exponent $\alpha=1+2/h$. For smooth flows ($h=1$), one obtains $\alpha=3$ as previously derived [18]. During the large loops, the trajectories equally reach large *positive* values of X and of Y. Again the fraction of time spent at both X and Y larger than $x\gg 1$ can be estimated as $x^{-1/h}$. Hence, the PDF of both X and Y have algebraic left and right tails.

As shown in Fig. 3, the presence of power-law tails in the PDF is confirmed numerically, with perfect agreement between the measured values of α and the prediction $\alpha=1+2/h$ (see inset). Let us comment on the h dependence of α. The probability to enter large loops, which correspond to events in which particles approach each other very closely (i.e., the mechanism at the basis of particle clustering), decreases significantly when $h\to 0$. Moreover, it is straightforward to check from Eqs. (4)–(6) that during the loops $Z(s)\propto Z_0^h$ when $Z_0\ll 1$. Hence it gets less and less probable to reach smaller values of Z as h decreases. Combined together, these two effects explain why particle clustering is weakened in rough velocity fields and why it is more efficient in smooth flows.

The change of variables (7) can be equally applied in

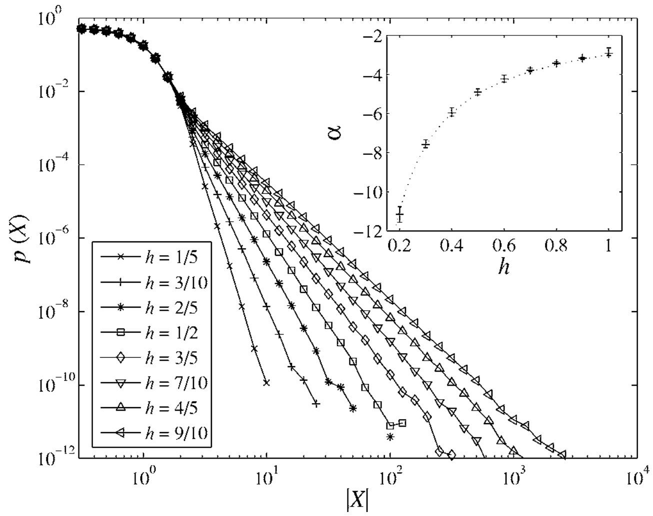

FIG. 3. PDF of X in log-log coordinates for $\mathcal{S}(L)=1$ and various values of h. In all cases, power-law tails are observed. Inset: exponent α of the algebraic tail as a function of the fluid velocity Hölder exponent h; the theoretical prediction is represented as a dotted line.

three dimensions, leading to a dynamics different from Eqs. (4)–(6). Therefore understanding to what extent the above findings extend to the three-dimensional case remains an open question; work in this direction is under development.

To conclude, let us comment on the implications of this work to the study of heavy particles in real turbulent flows. There, particle clustering is simultaneously due to ejection from eddies and to a dissipative dynamics. The considered model flow isolates the latter effect. It is probable that power-law tails for velocity differences can be present in realistic settings as well. However, it is not clear if the results on clustering are affected by the presence of persistent structures: particle ejection from eddies may form voids and thus very strong inhomogeneities in the particle distribution [11,13]. This could overtake dissipative-dynamics mechanisms. Another effect neglected in this study is the presence of gravity, which can be important in many realistic situations. Gravity provides a mechanism for the decorrelation of fluid velocity along particle paths. Therefore, including such an effect fits well in the time-uncorrelated model here discussed and represents a natural continuation of the present work.

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